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Coefficient of restitution mediated by a diamagnetic interaction

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Abstract. We study how a magnetic bead bounces onto a horizontal diamagnetic conducting plane. The bead, falling down by gravity from a certain height, produces an Eddy current that creates a repelling force. For low velocities the bead is trapped by the surface, for intermediate ones it escapes. In such a case the induced current changes its sign, and so does the force. The balance between diamagnetic and viscoelastic interactions determines the bouncing dynamics. We find experimentally the restitution coefficient as a function of the impact speed of the bead and develop, taking into account simple energetic considerations, a model able to reproduce our findings.

1 Introduction

During the transport and manipulation of granular materials, as well as in the study of dust conglomerates in planetary sciences, knowledge about the interaction forces between two or more particles is crucial to understand the dynamics of the involved processes [1]. Moreover, in order to perform efficient Molecular Dynamics Simulations we need to know how these interaction forces change with the impact velocity [2]. The coefficient of restitution (henceforth called simply e, and defined as the ratio of two velocities: after and before the impact) is a parameter of great relevance which gives a measure of the energy loss during the collision.

More than one hundred years ago H. Hertz perceived that the force acting during the contact between two spheres is repulsive and due to elastic deformation [3]. Thereafter, many authors have proposed different theoretical models to consider not only repulsion but adhesion [4–7], viscoelastic interactions [1, 8], coagulation [9], the influence of the duration of the interaction on the velocity dependence of the restitution coefficient [10], or high inelastic interactions [11], among several others. Recently, in refs. [12,13] the quasi-static approximation of viscoelasticity of ref. [8] has been extended in a rigorous mathematical framework to consider situations in which this approximation is not valid, and applied it to the interaction of bodies of different shapes (considered convex) and of different materials.

The aim of the present article is to report an experiment dealing with a different type of situation: a *soft* collision mediated by a diamagnetic force. A magnetic bead is released, from a given height, onto a diamagnetic plane. As the bead approaches the plane an Eddy current forms in it. The induced current produces a magnetic field that opposes the entrance of the bead's field. If the bead bounces (because for some conditions it is trapped) the Eddy current changes in sign and now opposes the bead's separation. Altogether, energy is dissipated not only by the hard-wall interaction where deformation occurs, but also by the magnetic interaction between the bead and the plane: Eddy currents dissipate energy by Joule's effect. A peculiar behavior is observed: first e is zero for small velocities, then it increases for moderate velocities, saturates and finally decreases for higher velocities. We carry out experiments with demagnetized beads to compare both coefficients and draw some conclusions.

The paper is organized as follows: sect. 2 is devoted to review the experimental details and show our results, sect. 3 develops theoretical estimations for the coefficient of restitution (with two appendices at the end), sect. 4 presents the discussion and finally, sect. 5 gives the conclusions.

2 Experiments and results

2.1 Experimental setup and details

A group of spherical magnets (SupermagnetM, NdFeB, covered with a hard epoxy layer) were subjected to thermal treatment at a temperature slightly above the Curie temperature (400 °C). Regulating the time in the oven, different magnetizations were obtained. We characterized

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the magnetizations by the maximum surface magnetic field, measured with a Gaussmeter (Lake Shore 410). Demagnetization times were chosen to obtain surface fields ranging from 0 to 12 kG. The procedure is similar to that reported in [14].

In the first experiment we tested the integrity of the beads. In other words, we wanted to know if the beads submitted to thermal treatment have changed or not their mechanical properties. For such purpose, control experiments on wood and Plexiglas surfaces were performed, comparing the coefficient of restitution of beads with and without thermal treatment. Only those beads having an average e_{control} statistically equal to the one for untreated beads were used in the following experiments.

The second experiment is dedicated to determine e for the beads impacting onto a horizontal copper surface. From a given height h a bead of (5.30 ± 0.02) mm of diameter, and a mass of $(0.495 \pm 0.004)g$ falls perpendicularly to the surface, while a CCD fast speed camera records its trajectory. Different heights give different velocities, from 0.10 to 5.50 m/s. The copper surface was mechanically polished (polishing tracks are visible only under optical magnification). We used a magnetic mechanism that releases the beads with their magnetic moment approximately parallel to their trajectory. Control experiments with marked beads confirmed that the beads fall almost with no rotation, so the impacts are mostly with the dipole perpendicular to the plane. Below we discuss the effect provoked by the deviations from such perpendicularity on the magnetic force strength. Care was taken to change periodically the beads and the point of impact in the plane, to avoid indentation and the deformation of the epoxy layer (which is around 200 μm), particularly for the higher velocities.

Films were digitized with the free software ImageJ, obtaining position *vs.* time data. The data are used to determine the speed of the bead by numeric derivation, and finally the coefficient of restitution is calculated as v_a/v_b, where the subscripts b and a mean before and after the impact (in what follows we will keep this nomenclature). As the magnetic interaction begins before the bead and the plane get into contact, and ends after their separation, it must be selected a point of beginning and a point of ending of the interaction. We choose, for reasons that will be discussed below, the points of imminent contact and separation of the beads. In order to obtain good resolution recording rates from 5000 to 15000 fps are used. From three to ten experiments per height and per magnetization were performed and their results averaged.

2.2 Experimental results

Figure 1 shows the experimental velocity dependence of e for a fully magnetized bead (12 kG, circles) and a totally demagnetized one (0 kG, squares).

Let us discuss first the results for magnetized beads. In the zone of low velocities the coefficient of restitution is zero. This effect is the first sign that a diamagnetic interaction dominates the collision: the bead does not have enough kinetic energy to escape from the plane because is trapped by a diamagnetic force that changes from repulsive to attractive (attractive once the bead stops and tries to bounce). Here, it is convenient to define the limit velocity v_l as the highest incoming velocity v_b for which the bead still has a bouncing speed $v_a = 0$. Furthermore, note that this trapping effect is not related to adhesion or surface energy [4] as we will discuss below.

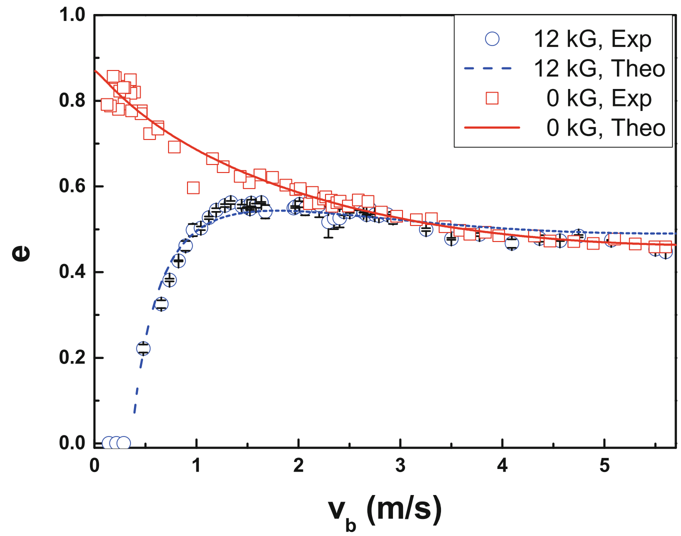

Fig. 1. (Color online) Velocity dependence of the coefficient of restitution for magnetized and fully demagnetized beads. The dashed line is a fit to eq. (10) while the continuous one is a fit to eq. (11), see text.

The value of v_l is around 0.32 m/s. For values of v_b above this threshold, the value of e increases until, around 1.50 m/s, levels off. It is easy to see a decrement of e for $v_b > 2.00$ m/s. Later in the paper we will explain the meaning of the dashed line that fits the data for all velocities.

We consider now the experiments using demagnetized beads. As fig. 1 shows, e monotonically decreases in all the velocity interval considered in the experiments. Again, we will later explain the nature of the continuous line fitting the data, although it is worth to advance that demagnetized beads rebound in such a way that viscoelastic interactions, at least for velocities up to 5.50 m/s, now mediate the collision.

The most striking feature of this figure is that, unexpectedly, above 2.00 m/s both data merge. In other words, the coefficients of restitution are the same.

Before we try to lay down a model to understand the nature of the above results, we show in fig. 2 the coefficient of restitution for beads with different magnetizations. The same trend is seen regardless the value of the bead magnetization: no rebound at low velocities, then an increment of e, a leveling of its values and a final decrease, similar to that of the demagnetized bead. In the inset of the figure the low-velocity part of the fittings to the experimental data is shown, which will be discussed in the next section. The continuous monotonically decreasing line represents the viscoelastic dependence of the demagnetized

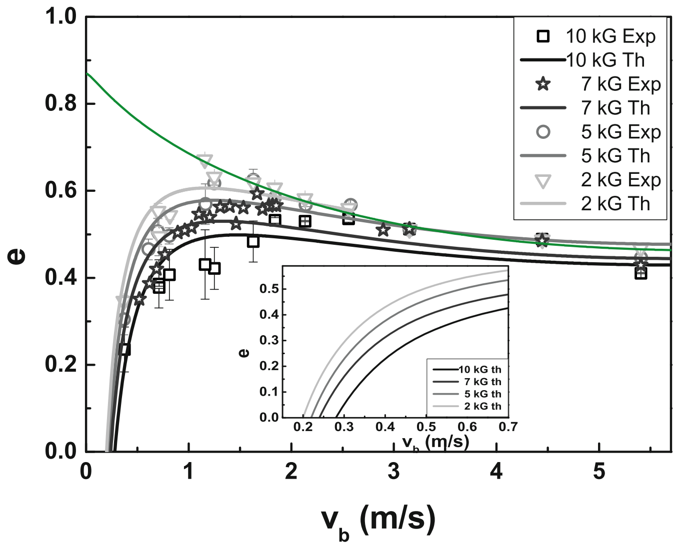

Fig. 2. Velocity dependence of the coefficient of restitution for partially demagnetized beads. In each case the solid line represents a fit of the model. The decreasing curve (green) corresponds to the demagnetized (0 kG) beads as in fig. 1. The inset shows the low velocity part of the fits.

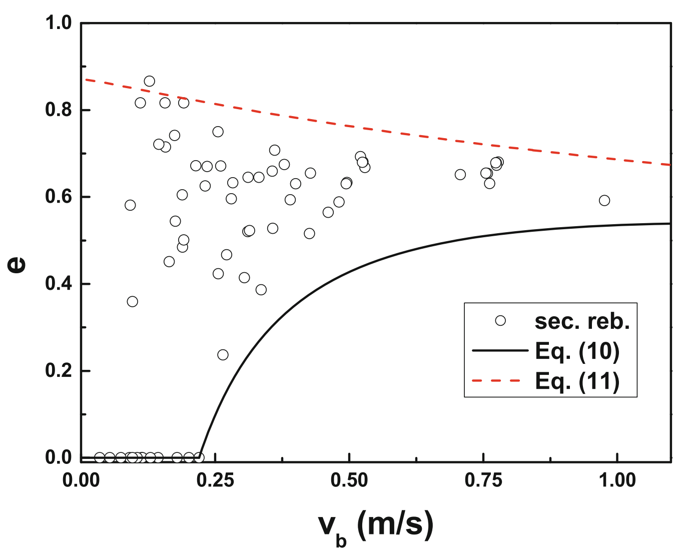

Fig. 3. Velocity dependence of the experimental coefficient of restitution for secondary rebounds of partially demagnetized beads of 5 kG (circles). The continuous line is a representation of eq. (10) for those beads, and the dashed line is the viscoelastic dependence, eq. (11).

beads seen in fig. 1. It is possible to note that also for the partially magnetized beads, this behavior is dominant for velocities above 2.5 m/s.

The method used to obtain the limit velocity v_l is to film not only the first bounce, but also several secondary rebounds, as is shown in fig. 3. This, of course, would give rebounds with a variety of values of e, among which, we choose only those with $v_a = 0$. We will discuss further on this subject.

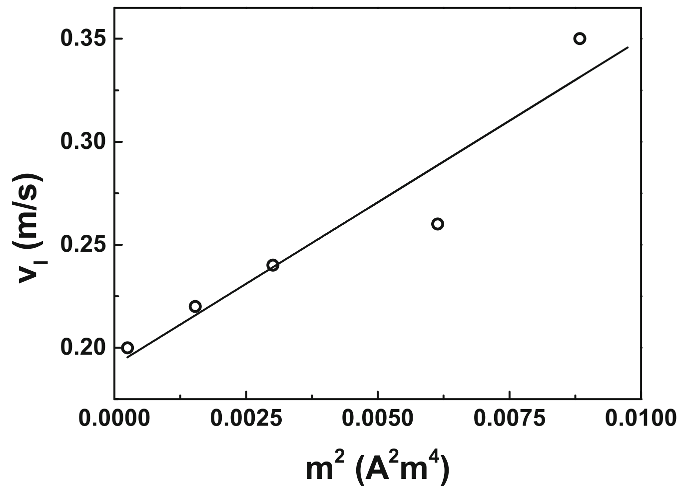

Fig. 4. The limit velocity v_l as a function of the square of the bead magnetization. The continuous line is the fit of a linear model.

The values of v_l determined as explained are plotted in fig. 4 as a function of the square of the magnetization of the bead m^2. A clear linear $v_l(m)$ dependence is observed.

3 Model

3.1 The magnetic force

The system under consideration is depicted in fig. 5. A magnetic bead, with its dipolar moment $\boldsymbol{m}$ pointing downward, falls under the action of gravity g along the Z axis of a reference frame and impinges on a conducting diamagnetic disk of radius ρ_0 containing the X and Y axis. The magnetic moment is perpendicular to the surface of the disk.

As the bead approaches the plane, it induces an Eddy current which in turn produces an opposing magnetic moment $\boldsymbol{m}_{\text{ind}}$ (Lenz's law). Between both moments there is a force that can be calculated with the following equation (the first term for the dipole current model, the second for the magnetic charge one) [15]:

$$\boldsymbol{F} = \nabla(\boldsymbol{m} \cdot \boldsymbol{B}_{\text{ind}}) = (\boldsymbol{m} \cdot \nabla)\boldsymbol{B}_{\text{ind}}. \qquad (1)$$

where $\boldsymbol{B}_{\text{ind}}$ is the magnetic induction of the field of the magnetic bead.

Independently of the model used the force between the magnetic moments is [16]

$$\boldsymbol{F} = \frac{\mu_0 m m_{\text{ind}}}{4\pi d^4}\big\{\hat{d}(\hat{m}_{\text{ind}} \cdot \hat{m}) + \hat{m}_{\text{ind}}(\hat{d} \cdot \hat{m}) + \hat{m}(\hat{d} \cdot \hat{m}_{\text{ind}}) - 5\hat{d}(\hat{d} \cdot \hat{m}_{\text{ind}})(\hat{d} \cdot \hat{m})\big\}, \qquad (2)$$

where m_{ind} is the magnitude of the induced dipole moment due to the Eddy currents in the plane and d is the distance between the center of the bead and the center of the currents associated with the induced dipole moment. The hat represents, as usual, the unitary vector symbol.

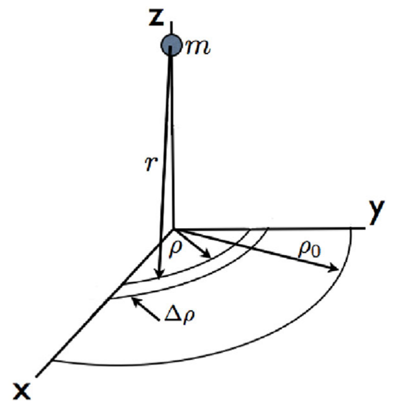

Fig. 5. (Color online) Schematic representation of the experimental system, used in the development of the model.

To calculate $\boldsymbol{m}_{\text{ind}}$ we have to determine the magnetic induction of the bead in the plane:

$$\boldsymbol{B} = \frac{\mu_0}{4\pi r^3}(3\hat{r}\hat{r}\cdot\boldsymbol{m} - \boldsymbol{m}), \tag{3}$$

where $\boldsymbol{m} = -m\hat{k}$ and r is the distance from the center of the magnetic bead to a point in the plane (see fig. 5).

The induced magnetic dipole moment is given by (see appendix A for details of calculation):

$$\boldsymbol{m}_{\text{ind}} = \frac{\mu_0 m}{4}\kappa f(\rho_0, z)z'\hat{k}, \tag{4}$$

where

$$f(\rho_0, z) = \frac{3z\rho_0^2 + 2z^3}{(\rho_0^2 + z^2)^{3/2}} - 2, \tag{5}$$

and z' is the magnitude of the velocity of the bead.

$f(\rho_0, z)$ is a function that approaches linearly to -2 as $z \to 0$. κ is a parameter of the material of the plane, ($\kappa = \sigma_{Cu}\xi$ is the product of the conductivity of the material, copper in our case, and the penetration length of the induced current). Introducing eq. (4) in eq. (2), we obtain the magnitude of the force acting on the bead when the dipole moment is oriented vertically:

$$\boldsymbol{F} = \frac{\mu_0^2 m^2}{8\pi z^4}\kappa f(\rho_0, z)z'\hat{k}. \tag{6}$$

It is easy to probe that $f_1(\rho_0, z) = f(\rho_0, z)z^{-4}$ has a pronounced dependence with z: $f_1 \sim z^{-4}$. Since $f(\rho_0, z)$ is always negative (see appendix A) and when the bead approaches the plane $z' < 0$, $\boldsymbol{F} \parallel \hat{k}$ agreeing with the prediction derived from Lenz's law. After bouncing, $z' > 0$, $f(\rho_0, z)$ keeps its sign, so now $\boldsymbol{F} \parallel -\hat{k}$, meaning that the force is attractive as is also predicted by Lenz's law. Of curse, f_1 is not capable to reproduce the behavior of the interaction force F for distances smaller than the bead's radius, where a monopole like behavior ($F \sim z^{-2}$) has been reported [17]. In any case (which has importance in what we do below) $f_1(\rho_0, z)$ is a monotonic increasing function of z. If it were not so, an equilibrium point would be reached, contradicting the physics of the problem.

3.2 Coefficient of restitution

In order to find an expression for the coefficient of restitution, let us first inquire about the forces involved in the interaction of the magnetic bead and the diamagnetic conducting plane. An important interaction, due both to its range and strength is, of course, the magnetic one. This interaction is highly non linear, due to its dependence on the distance between the sphere and the plane and on the velocity. Moreover, as discussed above, this interaction changes its sign after the impact.

Suppose the dipole of the bead remains perpendicular to the surface, then the magnetic force will increase as it approaches to the surface, because the velocity augments and the distance bead-surface reduces (see eq. (6)). But the increment of the magnetic force tends to slow the bead, therefore diminishing further increments of the speed. In addition to this complicated non-linear interaction there is another point to consider: the magnetic force is limited by the magnitude of the intensity of the induced current, which is ultimately related to the resistivity of the material. This makes this force hard to calculate in detail although we will give below an estimate of the energy dissipated due to its action. Furthermore, the problem gets even more entangled when we take into consideration that, as we mentioned before, there are different interaction regimes for the magnetic force: from z^{-4} (long-mid distance) to a monopole like interaction at short distances, smaller than the radius of the bead, with a distance dependence of the form z^{-2} [17].

Another important contribution is the elastic term. The elastic deformations are not permanent, meaning that when the interaction finishes the interacting bodies recover their original shape. Following ref. [18] the expression for the Hertzian force is $F_{el} = \rho\xi^{3/2}$, where ρ is determined by the elastic constants and dimension of the interacting bodies, and ξ is the total deformation of both surfaces.

Viscous losses appear due to the dissipation of energy in the bulk of the bodies, related with the change of deformation with time. For low velocities (compared with the speed of sound in the materials) and long collision times (compared with the characteristic time of the viscous processes), the viscous force can be computed as [18]: $F_{\text{vis}} = \frac{3}{2}A\rho\xi'\sqrt{\xi}$. In this expression A is determined from the viscous characteristic times and the elastic constants. As the viscous force increases with velocity, the fingerprint of its action is a monotonic decrease of the coefficient of restitution with velocity.

Finally, let us consider the possible influence of the so called surface adhesion [1,4,18]. This force is provoked by the molecular interactions between surfaces, and its effect has been mainly (but not only) studied for small velocities in soft and optically smooth surfaces. Adhesion becomes important when the distance of the particle surfaces approaches to the range of molecular forces [18]. Indeed, in the classical work of Johnson, Kendall and Roberts [4] authors used rubber and gelatin optically smooth surfaces of low Young modulus for measuring this force.

It is important to note, however, that in ref. [19] the authors measured kinematic coefficients of normal restitution in head-on collisions of two identical small spheres of acrylic, ceramic or steel at low impact speeds, and observed that adhesion lowers the coefficient of restitution. They found values of critical velocities ($e(v_{\rm crit}) = 0$) close to 0.02 m/s for normal impact between steel spheres. As noted very recently in ref. [20], the impact at small enough relative velocities between non-smooth ice particles results in adhesion, forming aggregates, which is important, for instance, in the particle size distribution of planetary disks.

In our experiments we worked at velocities not so small, been the limit velocities around 0.2 m/s, well above the highest experimental values of critical velocities obtained previously. So the velocity at which $v_a = 0$ must be associated to a different interaction. Indeed, it is reasonable to think that the surface interaction (adhesion) does not depend on the magnetization of the beads. If this contribution were important, it must appear in the interaction of demagnetized beads with the plane but, as fig. 1 shows, at velocities of 0.1 m/s adhesion is not perceptible. So, we will not consider its effect.

Now that we discussed the interactions to be considered, instead of solving the differential equation relating all these forces with the acceleration of the bead, we proceed with an energetic balance (using the theorem of work and energy) of the interactions to obtain a quantitative relation between e and v_b. Firstly it is necessary to determine which are the initial and final points in the movement of the bead, and consider the energy transformations in the transit from one point to the other.

In order to select the initial and final velocities used in the calculation of e, let discuss its definition first. In refs. [1,2], for instance, e is defined as the ratio of the velocities $\xi'(t_c)/\xi'(0)$ where $\xi(t)$ is the compression of the interacting surfaces. The interacting time t_c is defined in [2] as the time when the interaction force vanishes. The initial time is defined as the time "the spheres start contacting". It is useful to remember that in ref. [10] a correction is made, because the real interaction time is smaller than t_c as defined in [2]. When we consider forces that act not only during the contact (as is the case of the magnetic force that has, in principle, infinite range) we must do an *a priori* selection of the start and the end of the interaction.

The selection of the initial and final points in this kind of interaction has to be done with a certain degree of arbitrariness. Since there are several possibilities we are going to discuss only three of them:

1. To select a fixed distance sphere-plane, defined as that at which we can neglect the magnetic force (where the interaction begins and ends). Though it seems easy to do, it is far from obvious: as the incoming and outgoing velocities are different, the forces will be different at that distance, which means that at a distance where the force is negligible for the outgoing bead, it is not for the incoming one. Besides, this distance will depend on the velocity, provoking its variation between different experiments.
2. To select a constant value of the force: it takes us to a similar problem as the previous definition, because in the point of equal forces the potential energy in the incoming and outgoing parts of the movement would be different complicating hence the equations.
3. To consider a fixed distance, for which the forces are not negligible: this is the one we choose, taking a distance between the center of gravity of the bead and the diamagnetic plane equal to the radius of the bead, where the velocities are easy to determine experimentally.

In our choice, of course, we took into account the fact that no matter which distance we use, the definition will be always arbitrary. We make the choice of considering (as we said above) the initial and final points at the height where the center of the bead is at a distance from the plane equal to its radius. For doing this, we analyzed the information provided by the digitization of the videos: there is not an appreciable departure from the free falling until the bead does not enter in contact with the plane. To check this we solved numerically the differential equation of the trajectory for fully magnetized and demagnetized beads, and did not find differences between both trajectories at distances between the surfaces of the plane and the bead as small as 0.1 mm. This is related with the fact that the interaction has a short range, and also that the numerical coefficient in the equation of the force depends on $\mu_0^2 m^2$, that results a small quantity, so only at very short distances the magnetic force is important for the behavior of the particle.

With the selection we have made of the initial and final points, we will determine the coefficient of restitution. The kinetic energy of the bead at the initial point will be partially dissipated by the magnetic ($W_{\rm mb}$, the subindex b indicating before the impact) and viscous ($W_{\rm vb}$) interactions, and the rest will be stored as elastic energy ($E_{\rm elas}$), so

$$\frac{1}{2} m_p v_b^2 = E_{\rm elas} + W_{\rm mb} + W_{\rm vb}. \tag{7}$$

where m_p is the mass of the bead, and v_b, as we discussed above, is the velocity of the incoming bead in the point of imminent contact with the plane.

In the rebound stage, the elastic energy is partially lost doing work against the magnetic forces ($W_{\rm ma}$, in this case, attractive) and the viscous dissipation $W_{\rm va}$. The rest is transformed into kinetic energy, associated with the velocity v_a in the point where the bead is in tangential contact with the plane:

$$E_{\rm elas} = W_{\rm ma} + W_{\rm va} + \frac{1}{2} m_p v_a^2. \tag{8}$$

Substituting (8) in (7) we obtain

$$\frac{1}{2} m_p v_b^2 = \frac{1}{2} m_p v_a^2 + (W_{\rm mb} + W_{\rm ma}) + (W_{\rm vb} + W_{\rm va}). \tag{9}$$

In appendix B we estimate the energies W_m, W_v, and obtain the dependence of the coefficient of restitution with the initial velocity:

$$e(v_b) = e_0 e_{\mathrm{ve}}(v_b)\left(1 - \frac{v_l}{v_b}\right), \tag{10}$$

where e_0 is a free parameter and $e_{\mathrm{ve}}(v_b)$ is the coefficient of restitution of a demagnetized bead at the incoming velocity (associated with the viscous dissipation). The term in parenthesis in eq. (10) is mainly related with the magnetic interaction.

Due to the range of velocities used here [10]

$$e_{\mathrm{ve}}(v_b) = 1 + k_1 v_b^{1/5} + k_2 v_b^{2/5} + k_3 v_b^{3/5} + k_4 v_b^{4/5} + \dots. \tag{11}$$

As we saw before, eqs. (10) and (11) fit very well our experimental data.

4 Discussion

Before discussing the experimental results, it is very important to note that, for the fitting procedure, we first fit eq. (11) to the data of fig. 1 (non magnetized beads), considering only the first four coefficients of the expansion, and obtained the coefficients k_i. After that, these coefficients were used to fit eq. (10) to the data of fig. 1 (fully magnetized beads), so in this equation the fitting parameters are v_l and e_0. Note that as v_l could be evaluated from the experiments, e_0 is the only free parameter.

The results of the fit of the viscoelastic dependence to the data gives:

$$e_{\mathrm{ve}}(v_b) = 1 - 0.98 v_b^{1/5} + 2.69 v_b^{2/5} - 3.04 v_b^{3/5} + 1.02 v_b^{4/5}, \tag{12}$$

In the fit shown in fig. 1, we have to use the four terms of the expansion (eq. (12)) in order to obtain the correct signs of the coefficients as theoretically predicted in [18]. The fit is good enough to claim that for totally demagnetized beads the rebound is dominated by viscoelastic interactions, at least for velocities up to 5.60 m/s. Taking more terms in eq. (11) the result will be more accurate. For the range of velocities considered here, e decreases as predicted by this equation.

Of course, this type of interaction must be present also during the impact of magnetized beads. Indeed, when the bead arrives with a velocity equal or less than its limit velocity, it dissipates energy as heat due to the work done against the magnetic force and its own deformation. This implies that the maximum attainable coefficient of restitution should be equal to that of the demagnetized bead, justifying the form of eq. (10).

As the influence of the term in parenthesis in eq. (10) (associated with magnetic interactions) on the coefficient of restitution almost saturates ($e \approx 1$) for velocities of around ten times the limit velocity, above this limit the dominating influence will be that of the viscous dissipation, that increases with velocity. This explains the behavior observed in fig. 1.

Substituting eq. (12) in eq. (10), we are able to reproduce very well the data measured with magnetized beads plotted in fig. 1 for all the speeds above v_l. Not only the form of the experimental curve (firstly an increase, then almost constant, finally a slow decrease), but also the values of v_l and the constants of the model are obtained from the nonlinear fitting procedure. The value of e_0 obtained is 1.238.

The fitting procedure as described above was also applied to the data registered for partially demagnetized beads. For these beads all features observed for fully magnetized ones are also found (see fig. 2), although the limit velocity diminishes as the demagnetization increases. Above 2.00 m/s, all the curves follow the same trend where viscoelastic interactions dominate. This can be seen comparing with the monotonically decreasing curve corresponding to the demagnetized beads. The smaller the magnetization the faster the viscoelastic interactions dominate. For all these measurements, the average value of the free parameter e_0 is $\overline{e_0} = 1.0 \pm 0.2$. The values of e_0 are close to one, indicating that the approximations used to obtain eq. (10) are plausible.

To obtain the value of the limit velocity for each magnetization it is necessary to repeat the bouncing experiment several times, which can be done using secondary rebounds (as in fig. 3). Of course, it is impossible to know the orientation of the magnetic moment at each rebound but, if the experiment is repeated many times, there is a probability that some rebounds occur with the magnetic moment pointing to the surface, so the experimental points would be distributed in an area bounded, from below, by the line given by eq. (10). There is also the possibility that the bead knocks the surface with its magnetic moment parallel to it. In such a case e will be close to that of the demagnetized beads, which implies that the distribution of experimental points should be bounded, from above, by eq. (12). Figure 3 shows the results for beads with 5 kG, confirming our reasoning.

According to eq. (B.11) $e_{\mathrm{ve}}^2(v_l)v_l$ must vary linearly with the square of magnetization. Figure 4 shows an approximate linear dependence of v_l with m^2. The standard error of the slope and the intercept is around 10%, which is very good taking into account all the approximations used to obtain eq. (B.11), and the fact that the value of the coefficient of restitution is not considered. In fig. 6 we plot the dependence of v_l and of $e_{\mathrm{ve}}^2(v_l)v_l = v_{l,\mathrm{rect}}$ with m^2. The dashed line in this figure is a linear fit, with its intercept closer to zero, which is the value that should be expected according to eq. (B.11). The fact that the intercept is not zero could be attributed to two different facts. Firstly, the smaller is the magnetization, the more imprecise is the determination of the limit velocity, due to the weakness of the magnetic dipolar interaction, and the indirect method of determination. Secondly, it is possible that, for the smaller velocities, the dependence change it functional form, diminishing faster v_l with the magnetization.

Finally, let us discuss the reproducibility of the experimental results. Most of the results shown in figs. 1 and 2 are the average of three repetitions performed under the same experimental conditions but, for all magnetizations, control experiments with ten repetitions were performed

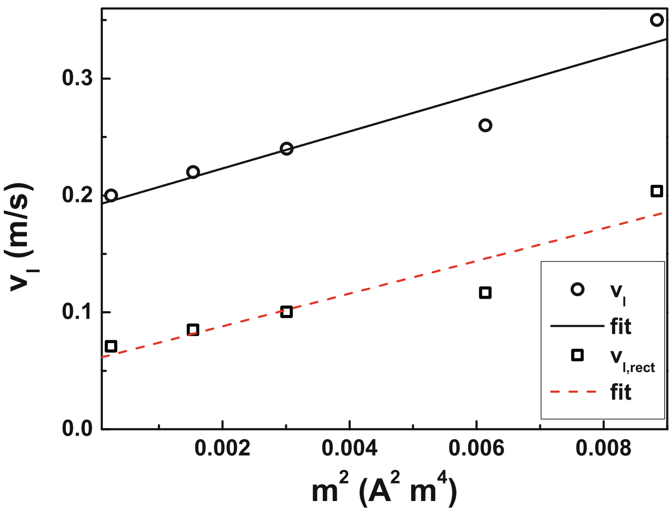

Fig. 6. Dependence of the limit velocity v_l (circles) and the rectified limit velocity $v_{l,\mathrm{rect}}$ (squares, see text for definition) with the square of the magnetization. The lines are linear fits.

at selected dropping heights, in order to test the reproducibility.

In most of cases the reproducibility is so good that the error bars Δe are smaller than the size of the symbols used. Only for the 10 kG beads, in the region of small and medium velocities the dispersion is such that the error bars are noticeable larger than the symbol's size.

An important discussion is the effect that a misalignment from the vertical direction of the magnetic moment of the falling bead has on the magnitude of the force. To do this we will refer to the vectorial part of eq. (2), because the other part does not change whit the change of orientation. If we compare the norm of this vector calculated for the magnetic moment pointing downward with the norm in the case of a misalignment by an angle ϕ we see that the values are 2 and $(1+3\cos^2\phi)^{1/2}$ respectively. It is easy to test that for angles smaller than 10° the relative error in the calculation is less that 6%. So a stochastic variation of the direction of the magnetic dipole will not affect significantly the magnetic force (and, of course, will not affect at all any other interaction present in the system), giving the same average value of the measured restitution coefficient. Of course, the higher the dropping point, the easier an important deviation from the perfect alignment will occur, but also the less important will be the magnetic component of the interaction (at velocities above 2 m/s the viscoelastic term dominates the dissipation, see fig. 1) and again this factor will not influence in an appreciable amount the coefficient of restitution.

5 Conclusions

Overall, we perform experiments to find the restitution coefficient of a magnetic bead that bounces onto a diamagnetic plane. Our main finding is that for low and moderate impact velocities, the coefficient of restitution is determined mainly by the magnetic interaction. This means that this interaction strongly dissipates the initial energy. For higher velocities the magnetic term saturates and the viscoelastic interaction takes over. A theoretical expression has been proposed to explain the obtained results. On the light of our simple model, it seems that magnetic and viscoelastic terms act separately depending on the incoming speed.

Our findings could be applied to understand systems where other interactions are present, like van der Waals adhesion as studied already in [21]. Also, our results could be useful in industrial processes where colliding of magnetic and diamagnetic surfaces occur, or to design diamagnetic dampers. Finally, the interplay of forces here analyzed could also help to understand the dynamics of formation or disintegration of aggregates in simulations of stellar clouds and planet formation.

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Appendix A. Calculation of the magnetic force

We will consider a diamagnetic disk with the dipole falling along its axis as represented in fig. 5. To calculate the induced dipolar moment we will follow the approach:

1. Calculate the flux surrounded by the thin loop of radius ρ and width $\Delta\rho$.
2. Determine the induced *emf* using Faraday's law.
3. Using Ohm's law determine the induced current in the loop.
4. Calculate the induced magnetic moment that is associated with the current.
5. Sum the induced moment by all the disk to determine the total magnetic moment.

To do this, we determine from eq. (3) the magnetic induction provoked by the falling dipole in the plane in one instant of time. The value of $\boldsymbol{r}$ is

$$\boldsymbol{r} = x\hat{i} + y\hat{j} - z(t)\hat{k}. \tag{A.1}$$

In what follows we will omit the time dependence of z, unless it were necessary for the analysis. Then:

$$\boldsymbol{B} = \frac{\mu_0 m}{4\pi}\left[\frac{3z}{(\rho^2+z^2)^{5/2}}(x\hat{i}+y\hat{j}) + \frac{\rho^2-2z^2}{(\rho^2+z^2)^{5/2}}\hat{k}\right]. \tag{A.2}$$

The magnetic flux inside the ring is

$$\Phi_B = \int_0^\rho \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} = \int_0^\rho \boldsymbol{B}\cdot\hat{k}2\pi\rho'\mathrm{d}\rho', \tag{A.3}$$

where $\rho = (x^2+y^2)^{1/2}$. Then,

$$\Phi_B = -\frac{\mu_0 m}{2}\frac{\rho^2}{(\rho^2+z^2)^{3/2}}\,. \tag{A.4}$$

As the magnetic flux depends on time through z, the variable flux will create a *emf* through the ring:

$$\varepsilon = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = -\frac{3\mu_0 m}{2}\frac{z\rho^2}{(\rho^2+z^2)^{5/2}}z'(t). \tag{A.5}$$

In this equation the prime indicates time derivative. This *emf* provokes the circulation of an induced electric current in the ring, with intensity:

$$\Delta I_{\mathrm{ind}} = \frac{\varepsilon}{\Delta R}, \tag{A.6}$$

being ΔR the electric resistance of the ring to the circulation of the current. Considering that it has a penetration depth ξ in the ring, and terming σ_{Cu} the conductivity of copper, we find

$$\Delta R = \frac{1}{\sigma_{Cu}}\frac{2\pi\rho}{\Delta\rho\xi}. \tag{A.7}$$

then,

$$\Delta I_{\mathrm{ind}} = -\frac{3\mu_0 m}{4\pi}\kappa g(\rho,z)z'(t)\Delta\rho, \tag{A.8}$$

being $\kappa = \sigma_{Cu}\xi$ and $g(\rho,z) = (z\rho)/(\rho^2+z^2)^{5/2}$.

The current circulating by the thin ring induces a magnetic moment, equal to

$$\Delta\boldsymbol{m}_{\mathrm{ind}} = -\frac{3\mu_0 m}{4}\kappa g(\rho,z)z'(t)\rho^2\Delta\rho\hat{k}. \tag{A.9}$$

For calculating the total induced moment in the disk, we divide it in rings of differential width $\mathrm{d}\rho$, and sum over all of them:

$$\boldsymbol{m}_{\mathrm{ind}} = -\frac{3\mu_0 m}{4}\kappa z'(t)\hat{k}\int_0^{\rho_0} g(\rho,z)\rho^2\mathrm{d}\rho, \tag{A.10}$$

$$\boldsymbol{m}_{\mathrm{ind}} = \frac{\mu_0 m}{4}\kappa z'(t)\hat{k} \times\left[\frac{3\rho_0^2 z+2z^3}{(\rho_0^2+z^2)^{3/2}}-2\right]. \tag{A.11}$$

This value must be substituted in eq. (2) to obtain the force acting on the bead. Considering that the dipolar moment of the bead always points downward, the force between the magnetic bead and the dipole is, finally,

$$\boldsymbol{F} = \frac{\mu_0^2 m^2}{8\pi z^4}\kappa\left[\frac{3\rho_0^2 z+2z^3}{(\rho_0^2+z^2)^{3/2}}-2\right]z'\hat{k}. \tag{A.12}$$

The magnitude surrounded by square braces in eq. (A.12) has only complex roots and is always negative. It is important for what is discussed in the main text.

Appendix B. Calculation of the coefficient of restitution

Equation (9) gives the energy balance of the bouncing process of one bead

$$\frac{1}{2}m_p v_b^2 = \frac{1}{2}m_p v_a^2 + W_{\mathrm{ma}} + W_{\mathrm{mb}} + W_{\mathrm{ve}}, \tag{B.1}$$

where we put in one symbol the energy dissipated due to the viscous losses

$$W_{\mathrm{ve}} = W_{\mathrm{vb}} + W_{\mathrm{va}}. \tag{B.2}$$

Using eq. (6) we can calculate W_b and W_a

$$W_b = \int_{r_1}^{r_2}\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = \int_{z_0}^{z_{\min}} F\mathrm{d}z. \tag{B.3}$$

$$W_b = Km^2\int_{z_0}^{z_{\min}} f_1(\rho_0,z)v(z)\mathrm{d}z. \tag{B.4}$$

In this last equation K accounts for the constants involved. z_0 is the distance between the center of the bead and the plane in the point of imminent contact (equal to the radius of the bead) and $z_{\min}$ is the distance between the center of the bead and the plane when the bead stops ($z_0 - z_{\min}$ is the maximum deformation of the bead). Though we do not know the functional dependence of $v(z)$, $f_1(\rho_0,z)$ is a continuous monotonic increasing function for $z > 0$, so applying the theorem of mean value

$$W_b = Km^2 v(z^*)\int_{z_0}^{z_{\min}} f_1(\rho_0,z)\mathrm{d}z; \quad z^*\in(z_{\min},z_0), \tag{B.5}$$

$$W_b = Km^2 v(z^*)I(z_{\min},z_0), \tag{B.6}$$

where $I(z_{\min},z_0)$ is the value of the integral in eq. (B.5).

This value is impossible to determine without knowing the exact distance dependence of the force which, as we stated before, changes with the decrement of the distance. But though the dependence can change from z^{-4} to z^{-2}, the monotony of the dependence remains, and so the validity of eq. (B.6). It is important to note that in what follows we do not use the value of $I(z_{\min},z_0)$.

Identically

$$W_a = Km^2 v(z^{**})I(z_{\min},z_0); \quad v(z^{**})\in(z_{\min},z_0). \tag{B.7}$$

It is very likely that the velocity of the bead decreases monotonically from the moment of the impact to the moment it stops (and increases monotonically in the backward movement). Then it is plausible to consider that the average velocity in eq. (B.6) is proportional to the velocity at the beginning of the impact, as well as the average velocity in eq. (B.7) is proportional to the velocity at the end of the impact. So we can write

$$\frac{v(z^*)}{v_b} = \frac{v(z^{**})}{v_a} = \zeta. \tag{B.8}$$

Then eq. (B.1) may be written as

$$\frac{1}{2}m_p v_b^2 = K\zeta m^2 I(z_{\min},z_0)(v_b+v_a) + \frac{1}{2}m_p v_a^2 + W_{\mathrm{ve}}. \tag{B.9}$$

An estimate of the viscous loses could be obtained introducing the velocity dependent viscoelastic coefficient of restitution $v_{\mathrm{ve}}(v_b)$

$$W_{\mathrm{ve}}(v_b) = \left(1-e_{\mathrm{ve}}^2(v_b)\right)\frac{1}{2}m_p v_b^2. \tag{B.10}$$

Substituting (B.10) in (B.9) and considering that the bead contacts the plane at the limit velocity $v_b = v_l$, which implies $v_a = 0$

$$e_{\mathrm{ve}}^2(v_l)v_l = \frac{2K\zeta m^2 I(z_{\min}, z_0)}{m_p}. \tag{B.11}$$

This equation predicts a linear relation of $e_{\mathrm{ve}}^2 v_l$ with m^2. From eq. (B.11) we obtain

$$K\zeta m^2 I(z_{\min}, z_0) = \frac{m_p}{2} e_{\mathrm{ve}}^2(v_l)v_l, \tag{B.12}$$

so equation (B.9) reduces to

$$e_{\mathrm{ve}}^2(v_b)\frac{1}{2}m_p v_b^2 = \frac{1}{2}m_p v_a^2 + \frac{m_p}{2}e_{\mathrm{ve}}^2(v_l)v_l(v_b + v_a). \tag{B.13}$$

Dividing by $\frac{1}{2}m_p v_b^2$ and substituting $e(v_b) = v_a/v_b$

$$e^2(v_b) + e_{\mathrm{ve}}^2(v_l)\frac{v_l}{v_b}e(v_b) + e_{\mathrm{ve}}^2(v_l)\frac{v_l}{v_b} - e_{\mathrm{ve}}^2(v_b) = 0. \tag{B.14}$$

Solving for $e(v_b)$

$$\begin{aligned} e(v_b) = &-\frac{1}{2}e_{\mathrm{ve}}^2(v_l)\frac{v_l}{v_b} \\ &+\sqrt{\frac{1}{4}e_{\mathrm{ve}}^4(v_l)\frac{v_l^2}{v_b^2} - e_{\mathrm{ve}}^2(v_l)\frac{v_l}{v_b} + e_{\mathrm{ve}}^2(v_b)}, \end{aligned} \tag{B.15}$$

where only the positive root is considered because e is a positive quantity. Considering that the coefficient of restitution of the demagnetized bead is close to one for small velocities, is possible to obtain

$$e(v_b) = -\frac{1}{2}e_{\mathrm{ve}}^2(v_l)\frac{v_l}{v_b} + e_{\mathrm{ve}}(v_b)\sqrt{\frac{1}{4}\frac{v_l^2}{v_b^2} - \frac{v_l}{v_b} + 1}, \tag{B.16}$$

which finally yields

$$e(v_b) = e_{\mathrm{ve}}(v_b)\left(1 - \frac{v_l}{v_b}\right). \tag{B.17}$$

In order to take into account the approximations assumed in the obtention of eq. (B.17), we introduce a free parameter to be obtained from the fitting processes

$$e(v_b) = e_0 e_{\mathrm{ve}}(v_b)\left(1 - \frac{v_l}{v_b}\right). \tag{B.18}$$

This equation predicts that for incoming velocities that are around ten times larger than the limit velocity, the coefficient of restitution will mostly depend on the viscous interactions, so we can asume that the term in round braces in eq. (B.18) is mainly related with the magnetic interaction.

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